Der Beweis stützt sich auf eine Formel für $I_c(\rho_0)$. die wir hier nicht angeben.

### 3.3. Korollar

*Sei c eine geschlossene Geodätische ; $c^m$ bezeichne die m-fache Überlegung von c. Dann gilt :*

$$\text{Index } c^m = \sum_{\rho^m = 1} I_c(\rho)$$

*und die Funktion $\rho \in S^1 \to I_c(\rho)$ ist bis auf eine Konstante (die etwa durch das Theorem 1 festgelegt ist) bestimmt durch die Poincaré-Abbildung von c.*

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[4] W. KLINGENBERG – Der Indexsatz für geschlossene Geodätische. Math, Z. 139 (1974), 231.

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DISCUSSION

Pr **Marsden** – Are these formulas for $I_c(\rho)$ reflected by any complication in the formula in terms of conjugate points for closed geodesics ?

Pr **Klingenberg** – It does not seem so.

Pr **Voros** – I confirm that, by using the invariance properties of the Maslov index, it is possible to discard the geodesic nature of the flow and define the index of rotation of a closed orbit of a hamiltonian flow with an elliptic Poincaré map P (and probably also if P is the direct sum of a purely elliptic and a purely hyperbolic part).

Pr **Klingenberg** – D'accord.

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# SOME BASIC PROPERTIES OF INFINITE DIMENSIONAL HAMILTONIAN SYSTEMS

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RESUME

Nous considérons quelques propriétés fondamentales des systèmes hamiltoniens de dimension infinie. Les systèmes sont linéaires ou non linéaires. Par exemple, dans le cas des systèmes linéaires, nous démontrons une version symplectique du théorème de M. Stone. Pour les systèmes généraux, nous établissons les théorèmes de conservation de l'énergie et du moment. (Le moment d'un groupe dynamique a été introduit par B. Kostant et J.M. Souriau). Pour les systèmes de dimension infinie, ces lois de conservation sont plus délicates que dans le cas des systèmes de dimension finie, parce que les équations sont aux dérivées partielles.

ABSTRACT

We consider some fundamental properties of infinite dimensional Hamiltonian systems, both linear and nonlinear. For exemple, in the case of linear systems, we prove a symplectic version of the teorem of M. Stone. In the general case we establish conservation of energy and the moment function for system with symmetry. (The moment function was introduced by B. Kostant and J.M. Souriau). For infinite dimensional systems these conservation laws are more delicate than those for finite dimensional systems because we are dealing with partial as opposed to ordinary differential equations.

## INTRODUCTION

In this paper we prove a few theorems concerning infinite dimensional Hamiltonian systems. Further details and examples may be found in [3, 4, 7, 11, 12].

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It is evident that the notion of a Hamiltonian system plays a fundamental role in mathematical physics. One needs only to recall a few examples : classical mechanics, classical field theory, quantum mechanics, hydrodynamics of a perfect fluid, elasticity, and the dynamical aspects of general relativity. In view of this, it is useful to set down some of the fundamental properties of such systems, both linear and nonlinear.

After giving the basic definitions, we prove a symplectic version of Stones theorem, i.e. the basic existence theorem for linear Hamiltonian vector fields, and then we prove the basic conservation laws of mechanics in the presence of a symmetry group in the infinite dimensional case.

## 1. SYMPLECTIC STRUCTURES AND HAMILTONIAN SYSTEMS

**Strong and Weak Nondegenerate Bilinear Forms.**

Let $\mathcal{E}$ be a Banach space and $B : \mathcal{E} \times \mathcal{E} \to \mathbf{R}$ a continuous bilinear mapping. Then B induces a continuous linear map $B^b : \mathcal{E} \to \mathcal{E}^*$, $e \mapsto B^b(e)$ defined by $B^b(e) \cdot f = B(e, f)$. We call B *weakly nondegenerate* if $B^b$ is injective ; i.e. if $B(e, f) = 0$ for all $f \in \mathcal{E}$ then $e = 0$. We call B *nondegenerate* or *strongly nondegenerate* if $B^b$ is an isomorphism. By the open mapping theorem it follows that B is nondegenerate iff B is weakly nondegenerate and $B^b$ is onto.

If $\mathcal{E}$ is finite dimensional there is no difference between strong and weak nondegeneracy. However in infinite dimensions the distinction is important.

**Symplectic Forms**

Let P be a manifold modelled on a Banach space(*) $\mathcal{E}$. By a *symplectic form* we mean a two- form $\omega$ on P such that

a) $\omega$ is closed : $d\omega = 0$

b) for each $x \in P$, $\omega_x : T_xP \times T_xP \to \mathbf{R}$ is nondegenerate.

If $\omega_x$ in (b) is weakly nondegenerate, we call $\omega$ a *weak symplectic form.*

Darboux's theorem in the infinite dimensional case is due to J. Moser and A. Weinstein and is the following (the proof is given in Lang [8]).

*Let $\omega$ be a symplectic form on the Banach manifold* P. *For each* $x \in P$ *there is a local coordinate chart about* $x$ *in which* $\omega$ *is constant.*

**Corollary** – *If* P *is finite dimensional and* $\omega$ *is a symplectic form then*

a) P *is even dimensional, say* $\dim P = m = 2n$

b) *locally about each point there are coordinates*

$x^1, \ldots, x^n, y^1, \ldots, y^n$ *such that*

$$\omega = \sum_1^n dx^i \wedge dy^i$$

*Such coordinates are called* canonical.

For Darboux's theorem for weak symplectic forms, see Marsden [10] and Tromba [15].

**Hamiltonian Vector Fields**

Let N be a Banach manifold, with $D \subset N$. A *vector field with domain* D is a map $X : D \to T(N)$ such that, for all $x \in D$, $X(x)$ lies in $T_x(N)$, the tangent space to N at $x$. An *integral curve for* X is a map $c : ]a, b[ \subset \mathbf{R} \to D$ which is differentiable as a map into N and satisfies $c'(t) = X(c(t))$. A *flow for* X *is a flow* $F_t$ on D such that, for all $x \in D$, the map $t \mapsto F_t(x)$ is an integral curve of X. (Semi-flows and local flows for X are defined analogously).

A subset D of a Banach manifold N is a *manifold domain* provided

1) D is dense in N ;

2) D carries a Banach manifold structure of its own such that the inclusion $i : D \to N$ is smooth ;

3) for each $x$ in D, the linear map $T_x i : T_xD \to T_xN$ is a dense inclusion.

(The linear prototype of such a domain is a dense linear subspace D of a Banach space $\mathcal{E}$ such that D is complete relative to a norm stronger than that of $\mathcal{E}$).

**Definition** – Let $P, \omega$ be a weak symplectic manifold. A vector field $X : D \to TP$ with manifold domain D is *Hamiltonian* if there is a $C^1$ function $H : D \to \mathbf{R}$ such that

$$\omega_x(X(x), v) = dH(x) \cdot v \tag{1}$$

for $x \in D$, $v \in T_xD \subset T_xP$. (From this it follows that, for each $x \in D$, the linear functional $dH(x)$ on $T_xD$ extends to a bounded linear functional on $T_xP$).

As usual, we shall write $X_H$ for X.

Because $\omega$ is merely a weak symplectic form, there need not exist a vector field $X_H$ corresponding to every given H on D. Moreover, even if

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(*) See [8]. The tangent space to P at $x \in P$ is denoted $T_xP$.

H is a smooth function defined on all of P, $X_H$ in general will be defined only on a subset of P. It is, of course, uniquely determined by the condition (1) on the set where it is defined.

Here are two infinite dimensional examples (both linear).

a) *The Wave Equation*(*) : $P = H^1(\mathbf{R}^n) \times L_2(\mathbf{R}^n)$,

$$\omega((\phi, \dot\phi), (\psi, \dot\psi)) = \langle\dot\psi, \phi\rangle - \langle\dot\phi, \psi\rangle$$

where $\langle\ ,\ \rangle$ is the $L_2$-inner product, $D = H^2(\mathbf{R}^n) \times H^1(\mathbf{R}^n)$, and $H : P \to \mathbf{R}$ is defined by the formula

$$H(\phi, \dot\phi) = \frac{1}{2}\langle\dot\phi, \dot\phi\rangle + \frac{1}{2}\langle\nabla\phi, \nabla\phi\rangle.$$

The vector field $X_H : D \to P$ is given by

$$X_H(\phi, \dot\phi) = (\dot\phi, \Delta\phi)$$

b) *The Abstract Schrodinger Equation* : $P = \mathcal{H}$, a complex Hilbert space ; $\omega(\psi_1, \psi_2) = \operatorname{Im}\langle\psi_1, \psi_2\rangle$ ; $H_{op}$ a self-adjoint operator with domain $D \subset \mathcal{H}$ ; then

$$X_H(\phi) = -iH_{op}\,\phi$$

$$H(\phi) = \frac{1}{2}\langle H_{op}\,\phi, \phi\rangle.$$

Note that in (a), H is defined and smooth on all of P ; while in (b), H is defined and smooth only on D (equipped with the graph norm).

**Poisson Brackets**

If $X_f : D_1 \to TP$ and $X_g : D_2 \to TP$ are two Hamiltonian vector fields, we define the *Poisson bracket*

$$\{f, g\} : D_1 \cap D_2 \to \mathbf{R}$$

by

$$\{f, g\}(x) = \omega_x(X_f(x), X_g(x)).$$

Even in the linear case, it is very important to pay attention to domains of definition when trying to deduce global consequences from formal identities involving Poisson brackets. However, the following result-which is a trivial consequence of the definitions-shows that there is no problem in deducing conservation laws if the conserved quantity is everywhere defined.

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(*) $H^2$ denotes the Sobolev space (Yosida [16]).



*Let $X_H$ be a Hamiltonian vector field with domain* D. *Assume that $X_H$ has a $C^0$ flow $F_t : D \to D$. Let $f : P \to \mathbf{R}$ be a $C^1$ function, and suppose that*

$$\{H, f\} \equiv df \cdot X_H = 0.$$

*Then $f \circ F_t = f$. That is, $f$ is constant on the trajectories of the flow of $X_H$.*

## 2. LINEAR HAMILTONIAN SYSTEMS

In this section we shall look at linear semigroup theory in a Hamiltonian setting. Thus let $\mathcal{E}$ be a Banach space (real or complex), and let $\omega : \mathcal{E} \times \mathcal{E} \to R$ be a symplectic bilinear form. Then $\omega$ determines a differential form $\Omega$ of degree two, as follows. Because we can identify the tangent space $T_x\mathcal{E}$ with $\mathcal{E}$ in a canonical way, we define

$$\Omega_x : T_x\mathcal{E} \times T_x\mathcal{E} \to \mathbf{R} \quad \text{by} \quad \Omega_x(e, f) = \omega(e, f).$$

Note that $d\Omega = 0$ because $\Omega_x$ is constant as a function of $x$. If $S : \mathcal{E} \to \mathcal{E}$ is a *linear* map, so that $D_x S = S$, we have

$$(S^*\Omega)_x(e, f) = \Omega_{Sx}(Se, Sf) = \omega(Se, Sf). \tag{1}$$

Hence S is symplectic (that is, $S^*\Omega = \Omega$) if and only if the bilinear form $\omega$ is invariant under S.

Now let $U_t = e^{tA}$ be a one-parameter group (or semigroup) with generator A. We know [16] that the domain $\mathcal{D}(A)$ is a dense linear subspace of $\mathcal{E}$. We may regard A as a (linear) *vector field* if we make the usual identification $Ax \in \mathcal{E} = T_x\mathcal{E}$.

**Theorem 1** – *Let $\mathcal{E}$ be a real Banach space and let $\omega$ be a weak symplectic form on $\mathcal{E}$, with $\Omega$ the corresponding differential 2-form. Let A generate a one-parameter group (or semigroup) $U_t$ on $\mathcal{E}$. Then the following are equivalent :*

i) A *is a locally Hamiltonian vector field*(*) : $i_A\Omega$ is closed

ii) A *is skew-symmetric with respect to* $\omega$ ; *that is,*

$$\omega(Ae, f) = -\omega(e, Af) \quad \text{for} \quad e, f \in \mathcal{D}(A) \tag{2}$$

iii) A *is globally Hamiltonian-with energy function*

$$H(e) = \frac{1}{2}\omega(Ae, e), \; e \in \mathcal{D}(A) \tag{3}$$

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(*) $i_A\Omega = A \lrcorner \Omega$ is the interior product of A with $\Omega$. It is a one form on $\mathcal{D}$ (A), and is defined in the usual way [1].

iv) $U_t$ *is symplectic : that is, as remarked above,* $U_t$ *preserves* $\omega$.

*Moreover, under these conditions energy is conserved* : $H \circ U_t = H$ *on* $\mathcal{D}(A)$.

*Proof* : (i) $\Leftrightarrow$ (ii) : Write $\alpha = i_A \Omega$. Thus if $x \in \mathcal{D}(A)$ we have

$$\alpha(x) \cdot f = \omega(Ax, f).$$

We say A is locally Hamiltonian iff $d\alpha = 0$. Now, by definition, if $e, f \in \mathcal{D}(A)$,

$$d\alpha(x) \cdot (e, f) = (D\alpha(x) \cdot e) f - (D\alpha(x) \cdot f) e$$

$$= \frac{d}{dt}\bigg|_{t=0} [\omega(A(x+te), f) - \omega(A(x+tf), e)]$$

$$= \omega(Ae, f) - \omega(Af, e).$$

Thus $d\alpha = 0 \Leftrightarrow$ A is skew-symmetric relative to $\omega$.

(ii) $\Rightarrow$ (iii) : Assuming (ii), we wish to show that $A = X_H$, that is, that $\alpha (= i_A \Omega) = dH$. But if $x, f \in \mathcal{D}(A)$ we compute

$$dH(x) \cdot f = \frac{d}{dt} H(x, + tf)|_{t=0} = \frac{d}{dt} \frac{1}{2} \omega(A(x+tf), x+tf)$$

$$= \frac{1}{2} \omega(Ax, f) + \frac{1}{2} \omega(Af, x)$$

$$= \frac{1}{2} \omega(Ax, f) - \frac{1}{2} \omega(x, Af) = \omega(Ax, f) \text{ by (ii)}$$

$$= (i_A \Omega)_x f.$$

(iii) $\Rightarrow$ (i) : If (iii) holds, $i_A \Omega = dH$. That $d(dH) = 0$ is clear.

(ii) $\Leftrightarrow$ (iv) : If $e, f \in \mathcal{D}(A)$ we have

$$\frac{d}{dt} \omega(U_t e, U_t f) = \omega(AU_t e, U_t f) + \omega(U_t e, AU_t f),$$

which vanishes if (ii) is true. Hence $\omega(U_t e, U_t f)$ is constant, that is, equal to $\omega(e, f)$. As $\mathcal{D}(A)$ is dense the same is true for all $e, f \in \mathcal{E}$. Conversely, if (iv) holds and $e, f \in \mathcal{D}(A)$, we have the relation

$$0 = \frac{d}{dt} \omega(U_t e, U_t f)|_{t=0} = \omega(Ae, f) + \omega(e, Af) ;$$

thus (ii) is true.



Finally, if A is Hamiltonian and $e \in \mathcal{D}(A)$, we have

$$H(U_t e) = \frac{1}{2} \omega(AU_t e, U_t e) = \frac{1}{2} \omega(U_t Ae, U_t e)$$

$$= \frac{1}{2} \omega(Ae, e) = H(e).$$

In the case of a group of isometries on Hilbert space, Stone's theorem implies that the generator is not merely skew-symmetric, but skew-adjoint. We turn now to the symplectic analogue of this fact.

**Theorem 2** – *Let* $\omega$ *be a weakly non-degenerate symplectic form on* $\mathcal{E}$. *Let* A *be the generator of a* group $U_t$ *of symplectic transformations on* $\mathcal{E}$. *Then* A *is skew-adjoint relative to* $\omega$.

*Note* : If B is any linear operator on $\mathcal{E}_0$ with dense domain $\mathcal{D}(B)$, we define the adjoint $B^\dagger$ of B relative to $\omega$ in the following way [16]. The domain of $B^\dagger$ is the set of all $f \in \mathcal{E}$ to which there corresponds a $g \in \mathcal{E}$ such that

$$\omega(Be, f) = \omega(e, g) \quad \text{for all} \quad e \in \mathcal{D}(B).$$

We write $g = B^\dagger f$. It is easy to see that $B^\dagger$ is a well-defined, closed linear operator.

*Proof of Theorem 2.* We assert that $A^\dagger = -A$. Because A is skew-symmetric we have $A^\dagger \supseteq -A$, For the opposite inclusion, suppose that $f \in \mathcal{D}(A^\dagger)$ with $A^\dagger f = g$. Then for any $e \in \mathcal{D}(A)$ we have

$$U_t e = e + \int_0^t AU_s e \, ds.$$

So

$$\omega(U_t e, f) = \omega(e, f) + \int_0^t \omega(AU_s e, f)\, ds$$

$$= \omega(e, f) + \int_0^t \omega(U_s e, g)\, ds.$$

Now $U_t$ is invertible and symplectic, so $U_t^\dagger = U_{-t}$. Thus

$$\omega(e, U_{-t} f) = \omega(e, f) + \int_0^t \omega(e, U_{-s} g)\, ds.$$

Because $\mathcal{D}(A)$ is dense and $\omega$ is weakly non-degenerate it follows that

$$U_{-t} f = f + \int_0^t U_{-s} g \, ds.$$

Accordingly, $f \in \mathcal{D}(A)$ and $-Af = g = A^{\dagger}f$.

Theorem 2 is the analogue of the "easy" half of Stone's theorem. It is natural to ask whether its *converse* is true. Unfortunately, this is definitely *not* the case(*). However this can be recovered as follows.

Let $\mathcal{E}$ be a real Banach space with a weakly nondegenerate skew form $\omega$. Let A be a *densely defined* linear operator, and suppose that $A^{\dagger} = -A$ ; that is, A is *skew-adjoint relative to* $\omega$. Define the "energy" inner product by

$$[e , f] = \omega(Ae , f) \tag{4}$$

for $e , f \in \mathcal{D}(A)$. Note that $[\cdot , \cdot]$ is a symmetric bilinear form. Suppose in addition the *energy is positive definite* in the sense that there is a a constant $c > 0$ with

$$[e , e] = \omega(Ae , e) \geqslant c \, \|e\|^2. \tag{5}$$

(Here $\|\cdot\|$ is the norm of $\mathcal{E}$). Then in particular $[\cdot , \cdot]$ is a positive definite inner product on $\mathcal{D}(A)$. Let $\mathcal{H}$ be the completion of $\mathcal{D}(A)$ with respect to this inner product. Then $\mathcal{H}$ is a Hilbert space, and the inclusion map $i : \mathcal{D}(A) \subset \mathcal{E}$ extends to a continuous map $i : \mathcal{H} \to \mathcal{E}$, because of (5). (Here we use the fact that $\mathcal{E}$ is complete).

**Lemma** – *The map $i : \mathcal{H} \to \mathcal{E}$ (defined above) is one-to-one. Thus $\mathcal{H}$ can be identified with a subspace of $\mathcal{E}$, with i the inclusion map.*

*Proof* – Suppose $x \in \mathcal{H}$ with $i(x) = 0$. We shall show that $x = 0$. Since $\mathcal{H}$ is the completion of $\mathcal{D}(A)$ with respect to the inner product (4), we can find a sequence $\{x_n\}_1^{\infty}$ in $\mathcal{D}(A)$ which is Cauchy relative to this inner product, and which converges to $x$ in $\mathcal{H}$. Also, as $n \to \infty$,

$$x_n = i(x_n) \to i(x) = 0. \tag{6}$$

Now $\mathcal{D}(A)$ is dense in $\mathcal{H}$. If $y \in \mathcal{D}(A)$ we have

$$[x , y] = \lim_{n\to\infty} [x_n , y] = \lim_{n\to\infty} \omega(Ax_n , y)$$

$$= - \lim_{n\to\infty} \omega(x_n , Ay) = 0 \text{ , by (6).}$$

Conclusion : $x = 0$, as claimed.

Let $A_1$ be the restriction of A to the domain

$$\mathcal{D}(A_1) = \{e \in \mathcal{D}(A) : Ae \in \mathcal{H}\}. \tag{7}$$

(*) For instance, consider the operator associated to the wave equation : $A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$ on $L_2 \times L_2$.



We can regard $A_1$ as an operator on $\mathcal{H}$.

**Theorem 3(*)** – *Under the condition stated above, $A_1$ is a skew-adjoint operator on $\mathcal{H}$. Accordingly it generates a one-parameter group $U_t$ of isometries on $\mathcal{H}$. This group preserves $\omega_1$, the restriction to $\mathcal{H}$ of the symplectic form $\omega$. (Moreover $A_1$ has a bounded inverse).*

*Proof* – First note that $A_1$ is skew-symmetric relative to the inner product of $\mathcal{H}$. Indeed if $e , f \in \mathcal{D}(A_1)$ we have

$$[e , A_1 f] = [e , Af]$$

$$= \omega(Ae , Af) = - \omega(Af , Ae)$$

$$= - [f , A_1 e] = - [A_1 e , f].$$

Let $j : \mathcal{E} \to \mathcal{H}^*$ be the adjoint of $i : \mathcal{H} \to \mathcal{E}$ relative to $\omega$. That is, if $e \in \mathcal{E}$ and $x \in \mathcal{H}$ we define

$$(je)x = \omega(e , ix) = \omega(e , x). \tag{8}$$

Now if $y \in \mathcal{D}(A) \subset \mathcal{H}$ define $\hat{A}y = jAiy$. We have, then,

$$(\hat{A}y)x = \omega(Aiy , ix) = \omega(Ay , x) = [y , x]. \tag{9}$$

In other words, if $y \in \mathcal{D}(A)$ then $\hat{A}y = \theta y$ where $\theta : \mathcal{H} \to \mathcal{H}^*$ is the canonical map identifying a Hilbert space with its dual.

Suppose now that $e \in \mathcal{E}$ ; then $\theta x = je$ for some $x \in \mathcal{H}$. Thus, if $y \in \mathcal{D}(A) \subseteq \mathcal{H}$ we have

$$[x , y] = (je)y = \omega(e , y)$$

and

$$[y , x] = \omega(Ay , x) = - \omega(x , Ay).$$

Thus for all $y \in \mathcal{D}(A)$ we have the relation

$$\omega(x , Ay) = - \omega(e , y).$$

Conclusion : since $A^{\dagger} = -A$, it follows that $x \in \mathcal{D}(A^{\dagger}) = \mathcal{D}(A)$ and $Ax = e$. In particular A maps $\mathcal{D}(A)$ *onto* $\mathcal{E}$.

Since A maps $\mathcal{D}(A)$ onto $\mathcal{E}$ it follows immediately that $A_1$ maps $\mathcal{D}(A_1)$ onto $\mathcal{H} \subseteq \mathcal{E}$.

But now we can show that $A_1$ is skew-adjoint. First, we check that $\mathcal{D}(A_1)$ is dense in $\mathcal{H}$. Suppose that $z \in \mathcal{D}(A_1)^{\perp}$. Then for all $x \in \mathcal{D}(A_1)$, $0 = [x , z]$. Now $z = A_1 y$ for some $y$. Hence $0 = [x , A_1 y] = - [A_1 x , y]$ for all $x \in \mathcal{D}(A_1)$. As $A_1$ is surjective, y must be 0.

(*) Unfortunately this result is false if we only assume $c = 0$. (See Chernoff-Marsden [3]).

Now suppose $u \in \mathcal{D}(A_1^*)$. Then for all $x \in \mathcal{D}(A_1)$ we have

$$[A_1 x , u] = [x , A_1^* u].$$

But $A_1^* u = - A_1 v$ for some $v \in \mathcal{D}(A_1)$. Hence, for $x \in \mathcal{D}(A_1)$,

$$[A_1 x , v] = - [x , A_1 v] = [x , A_1^* u] = [A_1 x , u].$$

Hence $v = u$, again because $A_1$ is surjective. That is, $u \in \mathcal{D}(A_1)$ and $A_1^* u = - A_1 u$. This completes the proof that $A_1$ is skew-adjoint.

Finally, we verify that the skew form $\omega_1$ is left invariant by $U_t = e^{tA_1}$. If $x , y \in \mathcal{D}(A_1)$ we have

$$\omega_1(A_1 x , y) = \omega(Ax , y) = [x , y] = [y , x]$$
$$= \omega_1(A_1 y , x) = - \omega_1(x , A_1 y).$$

Thus $A_1$ is skew-symmetric relative to $\omega_1$, and so Theorem 2 implies that $U_t$ leaves $\omega_1$ invariant.

*Remark* – Theorem 3 was motivated by the modern treatment of "Friedrichs extensions" in terms of so-called scales of Hilbert spaces.

**Poisson brackets and commutators**

Let $\mathcal{E}$ be a Banach space with skew form $\omega$. Let A and B be two skew-symmetric linear operators on $\mathcal{E}$, with corresponding energy functions $H_A$ and $H_B$, as in Theorem 1. There is an interesting *formal* relation between the Poisson bracket $\{H_A , H_B\}$ and the operator commutator $[A , B] = AB - BA$. (It is easy to check that $[A , B]$ is skew-symmetric if A and B are ; but in general $[A , B]$ will not be skew-adjoint, except in the trivial case when A and B are bounded. In fact, in general $[A , B]$ will not even be densely defined or closable).

*Let* A *and* B *be skew-symmetric relative to* $\omega$. *Then if* $x$ *is in the domain of* $[A , B]$ *we have the relation*

$$\{H_A , H_B\} (x) = H_{[A , B]}(x).$$

This is easy to check.

**Symmetry groups and conservation laws** (linear case)

As above, consider $\mathcal{E}$, equipped with a weak symplectic form $\omega$. Let A generate $U_t$, a group (or semigroup) of symplectic transformations. Also let B generate a *group* $V_t$ of symplectic transformations. Let $H_A$, $H_B$ be the corresponding energy functions.



**Theorem 4** – *Suppose that* $V_t$ *is a symmetry group of the energy* $H_A$ *in the following sense : each map* $V_t$ *leaves* $\mathcal{D}(A)$ *invariant, and* $H_A \circ V_t = H_A$. *Then* $H_B$ *is a constant of the motion ; that is,* $U_t$ *leaves* $\mathcal{D}(B)$ *invariant and* $H_B \circ U_t = H_B$. *Moreover, the flows* $U_t$ *and* $V_t$ *commute ; that is,* $U_s V_t = V_t U_s$ *for all* $s$, $t$.

One can give a straightforward proof of this result in the context of semigroup theory. However we shall prove a nonlinear generalization of it shortly.

*Note* : In order to conclude that the flows $U_s$ and $V_t$ commute, it is *not* enough to have $\{H_A , H_B\} = 0$, i.e. $[A , B] = 0$. In fact Nelson has given a well-known counter-example : two skew-adjoint operators A , B such that $[A , B] \equiv 0$ on $\mathcal{D}(AB) \cap \mathcal{D}(BA)$, but such that $e^{sA}$ and $e^{tB}$ do *not* commute. Thus the infinite dimensional case is much subtler than the finite dimensional case and it is well to be wary of reliance on formal calculations alone.

## 3. A GENERAL CONSERVATION THEOREM

In infinite dimensional systems, conservation laws require rather delicate handling. In most cases (as in example (b) above) the putatively conserved quantity $f$ is defined only on a dense subset of phase space. Moreover, formal calculations are usually not sufficient to imply the desired conclusions. A very simple example occurs in quantum mechanics : if H is a symmetric, but non-self-adjoint, operator then energy can "leak out" of the system. There are a number of rigorous general conservation theorems that can be established ; the following one seems to be optimal, since the conditions on the flow are mild. The main requirement is that $f$ and H have a common manifold domain of definition.

*Theorem 5* – *Let* P, $\omega$ *be a weak symplectic manifold. Let* $X_H : D \to TP$ *be a Hamiltonian vector field with manifold domain* D. *Assume that* $X_H$ *has a* $C^0$ *flow* $F_t : D \to D$. *Let* $f : D \to \mathbb{R}$ *be a* $C^1$ *function, and assume there is an associated Hamiltonian vector field* $X_f$, *a continuous map from* D *to* TP. *Then*

$$\frac{d}{dt} f \circ F_t = \{f , H\} \circ F_t \quad \text{on D.}$$

*In particular, if* $\{f , H\} = 0$ *then* $f \circ F_t = f$ *on* D.

The crux of the present theorem is that we do not know *a priori* that $f \circ F_t$ is differentiable in $t$, so that we can't simply apply the chain rule.

*Proof of Theorem 5* — Given $u_0 \in D$, we shall show that

$$\frac{d}{dt} f(F_t(u_0))|_{t=0} = \{f , H\}\,(u_0).$$

This will establish the theorem. Choose a local chart(*) so that $u_0 = 0$. Abbreviate $F_t(u_0)$ by $u_t$. Then from $df = i_{X_f}\omega$, we have the local formula

$$f(u_h) = f(0) + \int_0^1 \omega_{\tau u_h}(X_f(\tau u_h) , u_h)\, d\tau .$$

Hence

$$\frac{1}{h}\{f(u_h) - f(0)\} = \int_0^1 \omega_{\tau u_h}\Big(X_f(\tau u_h) , \frac{u_h}{h}\Big)\, d\tau .$$

Now, as $h \to 0$, $u_h \to u_0 = 0$ in the topology of D. Therefore, since $X_f : D \to TP$ is continuous, $X_f(\tau u_h) \to X_f(0) = X_f(u_0)$ uniformly for $0 \leqslant \tau \leqslant 1$. Also,

$$\frac{u_h}{h} = \frac{u_h - u_0}{h} \to X_H(u_0)$$

as $h \to 0$. Accordingly, the integrand $\omega_{\tau u_h}\Big(X_f(\tau u_h) , \frac{u_h}{h}\Big)$ converges uniformly to

$$\omega_0(X_f(u_0) , X_H(u_0)) = \omega_{u_0}(X_f(u_0) , X_H(u_0)) \quad \text{as} \quad h \to 0.$$

Thus

$$\begin{aligned}\lim_{h\to 0} \frac{1}{h}\{f(u_h) - f(u_0)\} &= \int_0^1 \omega_{u_0}(X_f(u_0) , X_H(u_0))\, d\tau \\ &= \omega_{u_0}(X_f(u_0) , X_H(u_0)) \\ &= \{f , H\}\,(u_0).\end{aligned}$$

*Remark* — The hypothesis that $X_H$ has a $C^0$ flow on D is not unreasonable. It will certainly hold (assyming that D and P are modelled on separable Banach spaces) provided that $X_H$ has a $C^0$ flow $F_t$ on P such that each $F_t$ maps D continuously into itself ; cf. Chernoff-Marsden [2]. In concrete examples this is very often the case.

The same argument yields the following.

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(*) To be perfectly honest, we are assuming here the existence of a local chart which *simultaneously* "flattens" D and P. The existence of such charts does not automatically follow from our definition of manifold domains. On the other hand, in many applications P and D will be linear spaces to begin with.



**Corollary** — *Let $H : D \subset P \to \mathbf{R}$ be defined and smooth on the manifold domain D, and let $X_H$ exist on D. Let $f : D_f \subset P \to \mathbf{R}$ be defined and smooth on the manifold domain $D_f$, and let $X_f$ be defined and continuous on $D_f$.*

*Suppose that $D_f \subset D$, and that $X_H$ has a flow $F_t$ which leaves $D_f$ invariant. Moreover, assume that for $x \in D_f$, the mapping $t \mapsto F_t(x) \in D_f$ is continuous.*

*If $\{f , H\} = 0$ on $D_f$, then $f \circ F_t = f$ on $D_f$.*

*Note* — We do not assume that the inclusion $D_f \subset D$ is continuous.

As a special case, we have conservation of energy.

**Theorem 6** — *Let P be a weak symplectic manifold. Let $H : D \subset P \to \mathbf{R}$ be defined and smooth on a manifold domain D, and let $X_H$ be defined and continuous on D. Suppose that $X_H$ has a flow $F_t$ on D, and that, for $x \in D$, the map $t \mapsto F_t(x) \in D$ is continuous. Then $H \circ F_t = H$ on D.*

In concrete situations one needs to know that the Hamiltonian H and the putatively conserved quantity $f$ have a suitable common domain of definition, as in theorem 5. We turn to this question next and begin with the following proposition.

**Proposition 7** — *Let P, $\omega$ be a weak symplectic manifold, $D \subset P$ a manifold domain, and $H : D \to \mathbf{R}$ a $C^1$ function. Assume that there is a Hamiltonian vector field $X_H : D \to TP$ for H, and that $X_H$ has a unique (local) flow $F_t : D \to D$.*

*Let $\Phi : P \to P$ be a symplectic $C^1$ diffeomorphism such that $\Phi(D) \subset D$ and $\Phi$ is $C^1$ on D. Finally, assume that $H \circ \Phi = H$.*

*Then $\Phi \circ F_t = F_t \circ \Phi$.*

*Proof* : Let $x \in D$. Since $\Phi$ is symplectic, we have the relation

$$\begin{aligned}\omega_x(X_H(x) , v) &= \omega_{\Phi(x)}(T\Phi(x)\cdot X_H(x) , T\Phi(x)\cdot v) \\ &= dH(x)\cdot v \\ &= d(H\circ\Phi^{-1})\circ T\Phi(x)\cdot v .\end{aligned}$$

Thus $\omega_{\Phi(x)}(T\Phi(x)\cdot X_H(x) , \omega) = d(H\circ\Phi^{-1})\cdot\omega$ for all $\omega \in T_{\Phi(x)}P$. Since $H\circ\Phi^{-1} = H$, we conclude that

$$X_H(\Phi(x)) = T\Phi(x)\cdot X_H(x). \qquad (1)$$

Now define $G_s = \Phi \circ F_s \circ \Phi^{-1}$. Then for $x \in D$ we have

$$\frac{d}{ds} G_s(x) = T\Phi \cdot X_H(F_s(\Phi^{-1}(x))$$

$$= X_H(\Phi(F_s(\Phi^{-1}(x))) \quad \text{by} \quad (1)$$

$$= X_H(G_s(x)).$$

It follows that $G_s$ is a flow for the vector field $X_H$. Since the flow of $X_H$ is unique, $G_s(x) = F_s(x)$.

We are now ready for our main conservation theorem.

**Theorem 8** – *Let* P, $\omega$ *be a weak symplectic manifold. Let* $X_H : D \to TP$ *be a Hamiltonian vector field with flow* $F_t$ *as in the hypothesis of Proposition 7. Assume in addition that* $F_t$ *is a* $C^0$ *flow on* D *and that each map* $F_t : D \to D$ *is* $C^1$.

*Let* $\Phi_t$ *be a flow of* $C^1$ *symplectic diffeomorphisms on* P. *Assume that each* $\Phi_t$ *satisfies the hypotheses of Proposition 7. Then, in particular,* $\Phi_t : D \to D$ *is a flow on* D. *Let* Y *be the generator of this flow, and assume that its domain* $D_Y$ *is dense in* D. *Moreover, assume that the graph of* Y *is a submanifold of* TD. *We equip* $D_Y$ *with the graph manifold structure.*

*Finally, suppose there is a* $C^1$ *function* $K : D_Y \to \mathbf{R}$ *such that* $Y = X_K$, *i.e.* Y *is the Hamiltonian vector field on* $D_Y$ *associated with* K.

*Conclusions :*

a) $F_t$ *leaves* $D_Y$ *invariant and gives a* $C^0$ *flow on* $D_Y$

b) $F_t \circ \Phi_s = \Phi_s \circ F_t$ *for all* $s$, $t$

c) $F \circ F_t = K$ *on* $D_Y$.

*Proof* – Conclusion (b) follows immediately from Proposition 7.

To prove (a) : Let $x$ be an element of $D_Y$. Because

$$\Phi_s(F_t(x)) = F_t(\Phi_s(s))$$

it follows that $s \mapsto \Phi_s(F_t(x))$ is differentiable relative to D ; here we use the hypothesis that $F_t : D \to D$ is $C^1$. Hence $F_t(D_Y) \subset D_Y$. Moreover, we have the relation

$$Y(F_t(x)) = TF_t(x) \cdot Y(x).$$

It follows that $F_t$ is continuous on $D_Y$ relative to the graph topology, so it induce a $C^0$ flow on $D_Y$.

From the relation $H(x) = H(\Phi_t(x))$ we deduce that, for $x \in D_Y$, $dH(x) \cdot Y(x)$ ; that is, $\{H , K\} = 0$ on $D_Y$. We can now apply Theorem 5 of § 3 to the flow $F_t$ on $D_Y$, concluding that $K \circ F_t = K$.



*Remarks* –

1) The strong smoothness hypothesis that $F_t$ is $C^1$ on D was needed only to establish (a). If (a) can be verified by other means(*) then we can drop this smoothness condition.

2) The above form of the conservation theorem is useful primarily because (a) is one of the conclusions, rather than one of the hypotheses. In practice, the symmetry group $\Phi_s$ will usually be given explicitly, while $F_t$ is known only implicitly as the flow of some differential equation. Accordingly it may be difficult to write down an explicit domain for K which is invariant under the flow $F_t$. This difficulty is avoided above.

3) In many applications $\Phi_s$ is linear. In such cases the hypotheses on the manifold structure of $D_Y$ will be satisfied automatically.

**Symmetry Groups on Tangent Bundles.**

As an example, we spell out the above result in the special case of a symmetry group acting on a tangent bundle.

Recall that the second tangent bundle $T(TM) = T^2M$ carries a canonical involution $s$ (see Godbillon [6]). In a local chart, $TM \simeq U \times \mathcal{E}$ where U is an open subset of $\mathcal{E}$ ; then $T^2M \simeq (U \times \mathcal{E}) \times (\mathcal{E} \times \mathcal{E})$, and $s$ is given by the formula $s(x , e ; e_1 , e_2) = (x , e_1 ; e , e_2)$

**Proposition 9** – *Let* M *be a weak Riemannian manifold. Equip* TM *with the associated weak symplectic form. Let* $\Phi_t$ *be a continuous flow of smooth mappings, each of which is an isometry of* M, *so that the tangent flow* $T\Phi_t$ *is symplectic.*

*Let* X *be the generator of* $\Phi_t$. *Suppose the graph of* X *is a submanifold of* TM. *Put on* $D_X$ *the associated manifold structure.*

*The generator* Y *of* $T\Phi_t$ *is an extension of* $s \circ TX$. *Assume* $Y = s \circ TX$. *Then* $\Gamma_Y = s(\Gamma_{TX})$, *so that the graph of* Y *is a submanifold of* $T^2M$, *and* $D_Y = TD_X$.

*Finally,* $Y = X_{P(X)}$ *where* $P(X) : D_Y = TD_X \to \mathbf{R}$ *is given by the formula* $P(X)(v_m) = \langle v_m , X(m) \rangle$.

This momentum function P(X) is a special case of the moment of a dynamical group introduced by Kostant and Souriau. See [14] and also [9], [13].

We now want to apply the conservation theorem 8.

**Theorem 10** – *Let* M *be a weak Riemannian manifold, as above. Let* $V : D_0 \subset M \to \mathbf{R}$ *be smooth on a manifold domain* $D_0$. *Let* D *be the* *restriction of* TM *to* $D_0$ *and construct* $X_E$ *on a domain* N *(the restriction of* $TD_0$ *to* $N_0 \subset D_0$*), where it exists and where*

$$E(v) = \frac{1}{2} <v , v> + V(x) , v \in T_x M .$$

*Suppose* $X_E$ *has a flow* $F_t : N \to N$ *which extends to a continuous flow of* $C^k$ *mappings of* D *to* D, $k \geq 1$.

*Let* $\Phi_t$ *be a continuous flow of smooth isometries of* $D_0$ *(relative to the metric obtained from* M*). The tangents thereby extend to symplectic diffeomorphisms of* D *to* D. *Suppose* $V \circ \Phi_t = V$. *Let* X *be the generator of* $\Phi_t$ *on* $D_0$, *and* Y *that of* $T\Phi_t$ *on* D. *Assume the graphs of* X *and* Y *are submanifolds.*

*Then*

a) $F_t = T\Phi_s = T\Phi_s \circ F_t$ *on* D,

b) $F_t$ *leaves* $D_Y$ *invariant*

c) $P(X) \circ F_t = P(X)$ *on* $D_Y$, *(and hence on* D *restricted to* $D_X$*).*

For further details and examples see Chernoff-Marsden [3, 4].

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## DISCUSSION

Pr **Bleuler** – 1) May I ask about the difficulties with respect to the *non-linear* cases. Are there counter exemples, i.e cases (with higher powers of the interaction) in which there are no solutions ?

2) May I also ask about the possibilites of *second quantization* e.g can the well-known results of Glimm and Jaffe be reproduced ?

Pr **Marsden** – 1) For the non linear wave equations the situation is not completely settled. For example Segal has shown global weak solutions exist (for positive interaction energies), but uniqueness is not known. Existence of strong solutions holds for short time always and global for $p = 3$, $n = 3$ or even $p$ (2, 4, $n = 3$) if the initial data is small enough.

2) Hopefully so, but those results are probably several years off.

Pr **Voros** – In the Glimm-Jaffe constructive field theory, the classical limit (Goldstone picture) predicts qualitative but not quantitative features (like anomalous critical exponents) of the quantum theory.

Pr **Raczka** – It was recently proved by Glassey that for a large class of non linear wave equations $(\Box + m^2)\varphi = \lambda\varphi^p$ ($p = 2$, 4 etc) the global solution does not exist even for very smooth initial conditions.

Pr **Marsden** – Yes, but I believe the initial data is not small in $H^1$ norm, at least for $n = 3$, $p = 2$.

Pr **Tarski** – With regard to the previous questions and remarks on constructive field theory, I would like to phrase the question of the applicability of the theory in this way to your examples you generally assumed a Hilbert space $L^2(R^n)$. In field theory one has a Fock space, which is a Hilbert space, but not of the above form. But I suppose that the particular form $L^2(R^n)$ is not necessary for most of the discussion – is this so ?

Pr **Marsden** – Yes. For example, in the Hamiltonian formulation of fluid mechanics the spaces $W^{s,p} = L^s_p$ are very useful.

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# DEFORMATIONS OF NON LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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RESUME

Dans cet article, nous examinons en quel sens la linéarisation d'un système d'équations aux dérivées partielles non linéaire approche le système complet. Nous appliquons ces idées à l'étude des déformations de l'équation de courbure scalaire et des équations d'Einstein en relativité générale, ainsi qu'à l'étude des ensembles de métriques riemanniennes à courbure scalaire donnée. On montre que ces systèmes sont linéairement stables sous des hypothèses très générales ; nous étudions aussi les cas exceptionnels d'instabilité linéaire.

ABSTRACT

In this article we examine in what sense the linearization of a system of nonlinear partial differential equations approximates the full nonlinear system. These ideas are applied to study the deformations of the scalar curvature equation and Einstein's equations of general relativity, as well as the set of metrics wirth prescribed scalar curvature. We show that these systems are linearization stable under general hypotheses ; in the exceptional cases of instability, we study the isolation of solutions.

## 0 – INTRODUCTION

Let M be a compact manifold, let X and Y be Banach manifolds of maps over M, such as spaces of tensor fields on M and let

$$\Phi : X \to Y$$

be a non-linear differential operator between X and Y ; we assume $\Phi$ itself is a differentiable map. Thus for given $y_0 \in Y$,

$$\Phi(x) = y_0 \qquad (1)$$

as an equation for $x \in X$, is a system of partial differential equations. If $x_0 \in X$ is a solution to (1), we will say that a differentiable curve $x(\lambda)$,